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LETTERS TO THE EDITOR

EFFECT OF NON-IDEAL BOUNDARY CONDITIONS ON THE VIBRATIONS OF CONTINUOUS SYSTEMS

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1. INTRODUCTION

In vibrations of continuous systems, types of support conditions are important and have direct effect on the solutions and natural frequencies. Boundary conditions of real systems are idealized by different types of supports such as simply supported, clamped, sliding, free, etc. The real system is modelled by choosing one of the nearest ideal boundary conditions. It is always assumed that those ideal conditions are satisfied exactly. However, small deviations from ideal conditions in real systems indeed occur. For example, a beam connected at its ends to rigid supports by pins is modelled using simply supported boundary conditions which require deflections and moments to be zero. However, the hole and pin assembly may have small gaps and/or friction which may introduce small deflections and/or moments at the ends. Similarly, a real built-in beam may have very small variations in deflection and/or slope. These types of boundary conditions with small deviations from the ideal conditions are defined here as non-ideal boundary conditions.

Non-ideal boundary conditions are modelled using perturbations. The idea is applied to two beam vibration problems; simply supported beam, sliding–clamped beam. Effect of non-ideal boundary conditions on the natural frequencies and mode shapes are examined for each case using the Lindstedt–Poincaré technique. Next, an axially moving string with non-ideal boundary conditions is discussed. Assuming small variations in deflections, it is found that in addition to changes in natural frequencies, amplitudes of vibration may also change, resulting in growth or decay of amplitudes depending on the modes of vibrations and on the axial velocity of the string. In this part, the Lindstedt–Poincaré technique is not suitable due to amplitude variations and the Method of Multiple Scales is employed instead, since it can account for those variations.

2. BEAM VIBRATIONS

In this section, two different sets of boundary conditions will be treated. Free vibrations of an Euler–Bernoulli beam in dimensionless quantities can be written as follows:

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^4} = 0, \tag{1}$$

where the deflection w, the time t and spatial variable x are all dimensionless quantities related to the corresponding dimensional ones as follows:

$$x = x^*/L, \qquad w = w^*/L, \qquad t = (1/L^2)\sqrt{EI/\rho A}\, t^*, \tag{2}$$

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where EI is the flexural rigidity, ρ is the density, L is the length and A is the cross-sectional area of the beam. A review of the vast literature on the topic is beyond the scope of this study. For some exact solutions of beams and comparison with approximate ones having different support conditions see references [1–5] for example.

2.1. SIMPLY SUPPORTED BEAM

Here one may assume that both of the boundary conditions are non-ideal or only one is non-ideal. To reduce the algebra only the right-hand-side boundary conditions will be taken as non-ideal; hence,

$$w(0, t) = 0, \quad \frac{\partial^2 w}{\partial x^2}(0, t) = 0, \quad w(1, t) = \varepsilon a(t), \quad \frac{\partial^2 w}{\partial x^2}(1, t) = \varepsilon b(t). \tag{3}$$

small deflections and moments are permitted as deviations from ideal conditions. ε is a small perturbation parameter.

Assuming a solution of the form

$$w(x, t) = (A \cos \omega t + B \sin \omega t)\, Y(x) \tag{4}$$

and requiring the time variations at the boundary to be of the same form, gives

$$Y^{\text{iv}} - \omega^2 Y = 0, \quad Y(0) = Y''(0) = 0, \quad Y(1) = \varepsilon a, \quad Y''(1) = \varepsilon b, \tag{5}$$

where a and b are constant amplitudes. Following the Lindstedt–Poincaré technique, the mode shapes and frequencies are expanded in perturbation series as

$$Y = Y_0 + \varepsilon Y_1 + \cdots, \qquad \omega = \omega_0 + \varepsilon \omega_1 + \cdots. \tag{6}$$

Substituting equations (6) into equations (5) and separating at each order, one has

$$O(1); \quad Y_0^{\text{iv}} - \omega_0^2 Y_0 = 0, \quad Y_0(0) = Y_0''(0) = Y_0(1) = Y_0''(1) = 0; \tag{7}$$

$$O(\varepsilon); \quad Y_1^{\text{iv}} - \omega_0^2 Y_1 = 2\omega_0 \omega_1 Y_0, \qquad Y_1(0) = Y_1''(0) = 0, \qquad Y_1(1) = a,\ Y_1''(1) = b. \tag{8}$$

The well-known solution at the first order is

$$Y_0(x) = \sqrt{2} \sin n\pi x, \quad \omega_0 = n^2\pi^2, \quad n = 1, 2, 3, \ldots, \tag{9}$$

where $Y_0(x)$ is normalized such that $\int_0^1 Y_0^2\, \mathrm{d}x = 1$.

At order ε, one seeks for a solvability condition. Since the homogenous problem has a non-trival solution, the non-homogenous problem has a solution only if a solvability condition is satisfied [6]. The solvability condition requires that

$$\omega_1 = \frac{\cos n\pi}{\sqrt{2}\, n\pi} (n^2\pi^2 a - b). \tag{10}$$

Substituting equation (10) into equations (8) and solving, one obtains the correction for the mode shapes

$$Y_1(x) = \frac{(an^2\pi^2 + b)}{2n^2\pi^2 \sinh n\pi} \sinh n\pi x + \frac{\cos n\pi}{2n^2\pi^2} (n^2\pi^2 a - b)\, x \cos n\pi x. \tag{11}$$

The final solution is

$$Y(x) = \sqrt{2}\sin n\pi x + \varepsilon\left\{\frac{(an^2\pi^2 + b)}{2n^2\pi^2 \sinh n\pi}\sinh n\pi x + \frac{\cos n\pi}{2n^2\pi^2}(n^2\pi^2 a - b)\, x\cos n\pi x\right\}. \quad (12)$$

The frequencies due to non-ideal boundary conditions are

$$\omega = n^2\pi^2 + \varepsilon\frac{\cos n\pi}{\sqrt{2}n\pi}(n^2\pi^2 a - b). \quad (13)$$

Depending on the relative amplitudes of variations in deflections and moments, and mode numbers, frequencies may increase or decrease. For the special case of $n^2\pi^2 a = b$, no change in frequency is observed. When one considers the first mode, displacement variations at the right end cause a decrease in frequency whereas moment variations cause an increase in frequency. For the second mode of vibration the reverse is true.

2.2. SLIDING–CLAMPED BEAM

Here to reduce the algebra, one may assume that the sliding boundary condition (i.e., displacements are allowed in the vertical direction) is ideal but the built-in boundary condition at the right-hand side is non-ideal. Hence,

$$Y'(0) = Y'''(0) = 0, \qquad Y(1) = \varepsilon a, \qquad Y'(1) = \varepsilon b. \quad (14)$$

Proceeding in a similar way as presented in the previous section, the frequencies and mode shapes are finally calculated as follows:

$$\omega = \beta^2 - \varepsilon(b + a\beta\tan\beta), \quad (15)$$

$$Y(x) = \frac{1}{\cos\beta}\left(\cos\beta x - \frac{\cos\beta}{\cosh\beta}\cosh\beta x\right) + \varepsilon\left\{\frac{a}{\cosh\beta}\cosh\beta x + x\frac{b + a\beta\tan\beta}{2\beta\cos\beta}\left(\sin\beta x + \frac{\cos\beta}{\cosh\beta}\sinh\beta x\right)\right\}, \quad (16)$$

where β satisfies the equation

$$\cos\beta\sinh\beta + \sin\beta\cosh\beta = 0. \quad (17)$$

TABLE 1

Comparison of the first five natural frequencies for the ideal ($\varepsilon = 0$) *and non-ideal* ($\varepsilon = 0{\cdot}1$, $a = b = 1$) *cases*

n	ω_0 (Ideal)	ω (Non-ideal)
1	5·59332	5·72568
2	30·22580	30·67557
3	74·63889	75·40282
4	138·79125	139·86936
5	222·68310	224·07534

Note that $\varepsilon = 0$ corresponds to the frequencies and mode shapes for ideal boundary conditions. The first five natural frequencies corresponding to the ideal and non-ideal cases are given in Table 1. Although the non-ideal frequencies increased for the specific choice of $a = b = 1$, this may not always be the case. For the first five frequencies $\tan\beta \approx -1$ and if b is selected such that $b > a\beta$, then a decrease in the frequencies may also be observed. Generally speaking, slope variations at the right end have a tendency of decreasing the frequencies, whereas displacement variations have a tendency of increasing them. Contrary to the simply supported case, these effects do not alter with the number of modes.

3. AXIALLY MOVING STRING VIBRATIONS

In this section, a simply supported string moving with a constant axial transport velocity is considered. At both ends non-ideal boundary conditions are assumed; that is, one allows small deflections at the ends. For more information on the general context of axially moving materials see review papers [7, 8] and some more recent publications [9–21].

The dimensionless equation of motion for the problem is

$$\frac{\partial^2 y}{\partial t^2} + 2v\,\frac{\partial^2 y}{\partial x \partial t} + (v^2 - 1)\,\frac{\partial^2 y}{\partial x^2} = 0, \tag{18}$$

where y is the transverse displacement, v is the axial transport velocity, x is the spatial and t is the time co-ordinate. The dimensionless quantities are related to the dimensional ones as follows:

$$y = y^*/L, \quad x = x^*/L, \quad t = t^*\sqrt{P/\rho A L^2}, \quad v = v^*/\sqrt{P/\rho A}, \tag{19}$$

where L is the length, P is the tension force and ρA is the mass per length of the beam.

The non-ideal boundary conditions are formulated as follows:

$$y(0,t) = \varepsilon\, a(t), \qquad y(1,t) = \varepsilon\, b(t). \tag{20}$$

Contrary to the beam problems, variations in displacements at the boundaries affect frequencies as well as amplitudes. Therefore the Lindstedt–Poincaré technique is not suitable for this problem. Instead, the Method of Multiple Scales [6] is employed in search of approximate solutions.

The following expansion is assumed:

$$y(x,t;\varepsilon) = y_0(x, T_0, T_1) + \varepsilon\, y_1(x, T_0, T_1), \tag{21}$$

where $T_0 = t$ is the fast time scale and $T_1 = \varepsilon t$ is the slow time scale. Time derivatives are defined as

$$\mathrm{d}/\mathrm{d}t = D_0 + \varepsilon D_1, \qquad \mathrm{d}^2/\mathrm{d}t^2 = D_0^2 + 2\varepsilon D_0 D_1, \tag{22}$$

where $D_n = \partial/\partial T_n$. Substituting the expansions into the original equation and boundary conditions and separating at each order of ε, one finally has

$$O(1): \quad \begin{aligned} & D_0^2 y_0 + 2vD_0 y_0' + (v^2 - 1)\, y_0'' = 0, \\ & y_0(0, T_0, T_1) = y_0(1, T_0, T_1) = 0, \end{aligned} \tag{23}$$

$$O(\varepsilon):\quad \begin{aligned} &D_0^2 y_1 + 2vD_0 y_1' + (v^2-1)\,y_1'' = -\,2D_0D_1y_0 - 2vD_1y_0',\\ &y_1(0, T_0, T_1) = a(T_0, T_1), \quad y_1(1, T_0, T_1) = b(T_0, T_1)\end{aligned} \tag{24}$$

where prime denotes derivative with respect to the spatial variable.

At the first order, the solution is

$$y_0 = A(T_1)\,e^{i\omega T_0}Y(x) + \bar{A}(T_1)\,e^{-i\omega T_0}\bar{Y}(x), \tag{25}$$

where the mode shapes and frequencies are

$$Y(x) = e^{in\pi vx}\sin n\pi x, \quad \omega = n\pi(1-v^2), \quad n = 1, 2, 3, \ldots. \tag{26}$$

At order ε, one inserts the order 1 solution (i.e., equation (25)) into the right-hand side of equation (24):

$$D_0^2 y_1 + 2vD_0y_1' + (v^2-1)\,y_1'' = -\,2i\omega D_1 A e^{i\omega T_0}Y - 2vD_1 A\,e^{i\omega T_0}Y' + \text{c.c.}, \tag{27}$$

where c.c. stands for complex conjugate of the preceding terms. Assuming a solution of the following form:

$$y_1 = \psi(x, T_1)\,e^{i\omega T_0} + \text{c.c.} \tag{28}$$

and substituting into equation (27) and boundary conditions in equation (24), one has

$$\begin{gathered} -\,\omega^2\psi + 2vi\omega\psi' + (v^2-1)\,\psi'' = -2i\omega D_1 A Y - 2vD_1 A Y',\\ \psi(0, T_1) = aA(T_1), \quad \psi(1, T_1) = bA(T_1). \end{gathered} \tag{29}$$

The solvability condition [6] requires

$$D_1 A = i(v^2-1)(a - b\cos n\pi\,e^{-in\pi v})\,A(T_1), \tag{30}$$

whose solution is

$$A(T_1) = Ce^{i(v^2-1)(a - b\cos n\pi\ e^{-in\pi v})T_1}, \tag{31}$$

where $C = (1/2)\,ce^{i\theta}$ is a complex constant. Substituting back equations (31) and (26) into equation (25), returning to the original time variable and simplifying, one finally obtains the approximate response as follows:

$$\begin{aligned} y(x, t) = ce^{\varepsilon[(1-v^2)b\cos n\pi\sin n\pi v]t}\cos\{[n\pi(1-v^2) + \varepsilon(1-v^2)(b\cos n\pi v\cos n\pi - a)]\,t\\ + n\pi vx + \theta\}\sin n\pi x, \end{aligned} \tag{32}$$

where c and θ are constants to be determined by initial conditions. The frequency due to non-ideal boundary conditions is

$$\omega_{n,i} = n\pi(1-v^2) + \varepsilon(1-v^2)\,[b\cos n\pi v\cos n\pi - a]. \tag{33}$$

The ideal ($\varepsilon = 0$) and non-ideal frequencies ($\varepsilon = 0{\cdot}1$, $a = b = 1$) are drawn as functions of the axial transport velocity in Figure 1. Although the non-ideal frequencies are always smaller than or equal to the corresponding ideal ones in the figure, depending on the sign of the bracket of equation (33), an increase may also be observed. The right-hand variations in displacements have a tendency of increasing the frequencies, whereas the left-hand variations have a tendency of decreasing them.

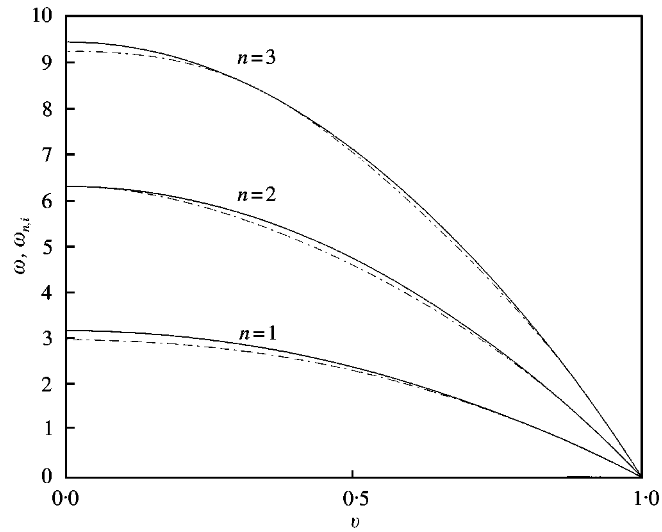

Figure 1. Ideal (——) and non-ideal (–·–·–·–) frequencies as functions of the axial transport velocity for first, second and third modes ($\varepsilon = 0{\cdot}1$, $a = 1$, $b = 1$).

TABLE 2

Qualitative behavior of amplitudes depending on the mode numbers and axial velocity

Mode numbers	Decay	Growth	Bounded
1	$0 < v < 1$	None	$v = 0$
2	$1/2 < v < 1$	$0 < v < 1/2$	$v = 0, 1/2$
3	$0 < v < 1/3$, $2/3 < v < 1$	$1/3 < v < 2/3$	$v = 0, 1/3, 2/3$
4	$1/4 < v < 1/2$, $3/4 < v < 1$	$0 < v < 1/4$, $1/2 < v < 3/4$	$v = 0, 1/4, 1/2, 3/4$
5	$0 < v < 1/5$, $2/5 < v < 3/5$, $4/5 < v < 1$	$1/5 < v < 2/5$, $3/5 < v < 4/5$	$v = 0, 1/5, 2/5, 3/5, 4/5$

Another interesting feature of the non-ideal boundary conditions is the growth or decay of amplitudes depending on the argument of the exponential term in equation (32). Since $(1 - v^2)$ and b are positive quantities, the sign of the exponential term is determined by the cosine and sine functions. The signs of these functions are determined by the mode number n and velocity v only. Depending on these parameters, the amplitudes of solutions may grow, decay or remain constant. For the first five modes, the qualitative behavior of amplitudes is summarized in Table 2.

The present analysis shows that the change of amplitudes results from the displacement variations at the right-hand-side boundary condition only. When $b = 0$, the frequencies are always lower and the amplitudes are always bounded in time.

The ideal ($\varepsilon = 0$) and non-ideal ($\varepsilon = 0{\cdot}1$) mode shapes are contrasted in Figure 2 for $v = 1/3$ for the first mode. The decay in amplitudes and the decrease in frequency are readily observed. For the second mode, one example plot for decaying amplitudes ($v = 2/3$) and another example plot for growing amplitudes ($v = 1/3$) are shown in Figures 3 and 4 respectively. In each of the figures, frequencies decrease due to non-ideal boundary

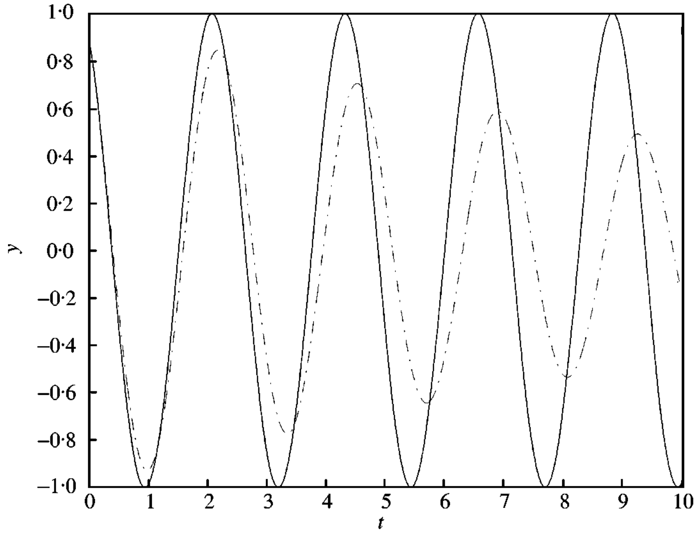

Figure 2. Comparison of vibrations of first mode for axially moving string for the cases of ideal boundary conditions (——) and non-ideal boundary conditions (–·–·–·–) ($\varepsilon = 0{\cdot}1$, $a = 1$, $b = 1$, $c = 1$, $\theta = 0$, $x = 1/2$ and $v = 1/3$).

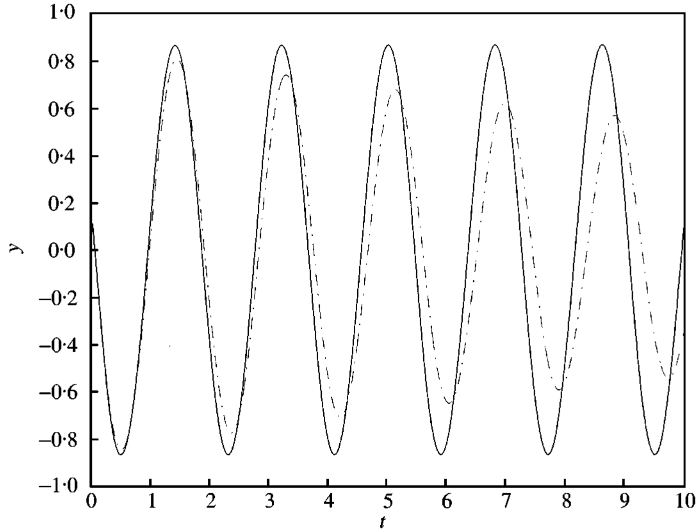

Figure 3. Comparison of vibrations of second mode for axially moving string for the cases of ideal boundary conditions (——) and non-ideal boundary conditions (–·–·–·–) ($\varepsilon = 0{\cdot}1$, $a = 1$, $b = 1$, $c = 1$, $\theta = 0$, $x = 1/3$ and $v = 2/3$).

conditions. However, if the values of b are increased further, an increase in frequencies may also occur.

4. CONCLUDING REMARKS

Non-ideal boundary conditions are defined and formulated using perturbation theory. Sample problems from the vibrations of continuous systems are treated. Two different beam

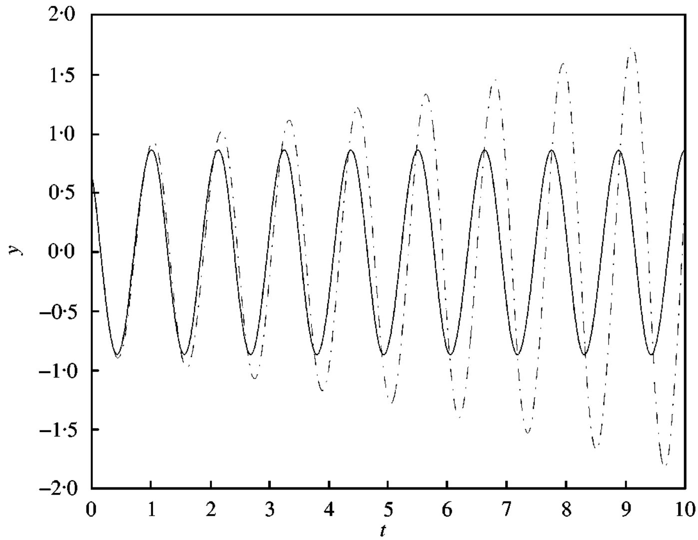

Figure 4. Comparison of vibrations of second mode for axially moving string for the cases of ideal boundary conditions (——) and non-ideal boundary conditions (–·–·–·–) ($\varepsilon = 0{\cdot}1$, $a = 1$, $b = 1$, $c = 1$, $\theta = 0$, $x = 1/3$ and $v = 1/3$).

vibration problems and an axially moving string problem are treated using the Lindstedt–Poincaré technique and the method of multiple scales. It is shown that non-ideal boundary conditions may affect the frequencies as well as amplitudes of vibration. Depending on the location of non-ideal support conditions and their small variations in time, frequencies may increase or decrease. For the beam problems, the effect is only on the frequencies. In the axially moving string problem, however, it is shown that in addition to frequencies, amplitudes of vibration also may grow or decay in time.

Linear vibrations of continuous systems are treated here using the non-ideal boundary condition concept. Non-linear vibrations of a beam using the same concept have been treated recently [22].

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